Lecture 11

Cosmic Inflation

Daniel Baumann has written an excellent set of lectures that you can find here: http://www.damtp.cam.ac.uk/user/db275/Cosmology.pdf

Also, Trodden's TASI lectures on cosmology are an excellent resource. See: https://arxiv.org/abs/astro-ph/0401547

1 Quick Recap

Using Einstein's field equations, which connect the geometrical properties of space-time to its energy content, and the FLRW metric one can derive the two independent Friedmann equations (see Lecture 5)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{2}$$

where ρ is energy density, p represents pressure, and G is Newton's gravitational constant. The equations describe interactions between a homogenous fluid and spacetime curvature in an FLRW universe and as such are applicable to the evolution of our universe on the largest scales. Equation 2 can be derived from the continuity equation (this is a useful exercise)

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{3}$$

Writing the equation of state for a perfect fluid as $p = w\rho$, we have already derived an equation for the scale factor in a flat universe:

$$a(t) = a_0 t^{\frac{2}{3(w+1)}}. \tag{4}$$

An ideal photon fluid is described by $w = 1/3$, whereas matter follows $w = 0$. The scale factor will therefore grow more rapidly in a matter dominated universe. From this one concludes that, being a mixture of photons, baryonic and dark matter, the universe will undergo phase transitions as different energy forms govern the development of the scale factor.

2 Horizons

Let's take a look at the propagation of light in the FLRW spacetime

$$ds^2 = a^2(\tau)\left(d\tau^2 - \frac{dr^2}{1 - kr^2} - r^2 d\Omega^2\right), \tag{5}$$

where we have transformed the time coordinate to conformal time defined by $d\tau = dt/a(t)$. Let's simplify the picture by assuming that the light ray is propagating in flat spacetime along a radial geodesic. This gives us

$$ds^2 = a^2(\tau)\left(d\tau^2 - dr^2\right). \tag{6}$$

Since light travels on null geodesics with $ds^2 = 0$, we get $\Delta\tau = \pm\Delta r$.

When we look at the cosmic microwave background (CMB), we are looking at an image of the universe from a redshift of $z \approx 1090$. The comoving distance traveled by a photo between times t_1 and t_2 is (in units where $c = 1$)

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{a(t)}. \tag{7}$$

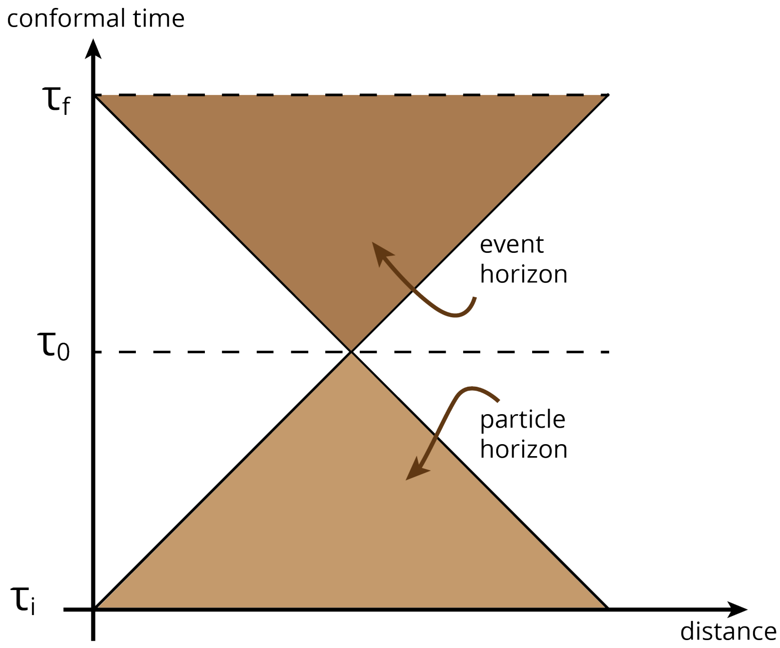

Figure 1: Particle and event horizons.

If we want to find the physical distance as measured by an observer at any time, t, we just need to multiply Δr with $a(t)$. By setting $t_1 = 0$, the above expression defines the maximum comoving distance that a signal traveling at the speed of light could traverse before arriving at an observer at time t_2. This is known as the **particle horizon** (see Figure 1).

Let's assume we want to find the maximum comoving distance that light could have traveled from time $t = 0$ to $t = t^{'}$ in a radiation dominated era $(a(t) = a_* \sqrt{t})$. From the equation above, it is clear that

$$\Delta r_{\rm rc} = \int_0^{t^{'}} \frac{dt}{a_* \sqrt{t}} = \frac{1}{a_*} 2\sqrt{t^{'}}. \tag{8}$$

This corresponds roughly to the particle horizon for an observer at the time of recombination. The corresponding physical distance is simply

$$d_{\rm rc} = \frac{a(t^{'})}{a_*} 2\sqrt{t^{'}} = \frac{a_* \sqrt{t^{'}}}{a_*} 2\sqrt{t^{'}} = 2t^{'}. \tag{9}$$

It turns out that this distance corresponds to about 1 degree on the sky. From

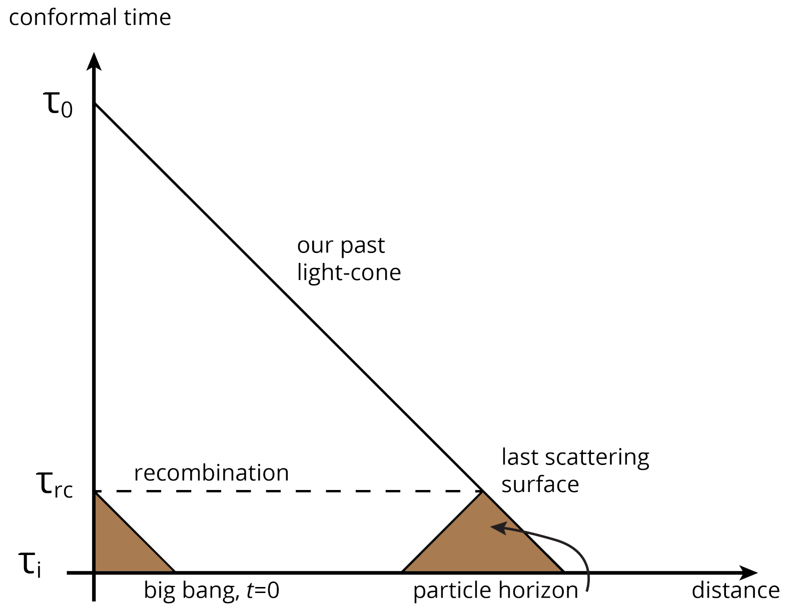

Figure 2: Conformal diagram showing the particle horizon of a photon that is emitted at recombination and arrives at our detectors at conformal time τ_0.

that, we conclude that the CMB is composed of multiple causally disconnected regions (see Figure 2). It seems unlikely that the universe could have agreed on such remarkable color coordination without at some point having been in causal contact. This is referred to as the **horizon problem**.

3 Flatness Problem

Another cosmological issue, a type of fine-tuning problem, can be discerned from rearranging terms in the first Friedmann equation (Eq. 1) and defining $\Omega = \rho/\rho_c$:

$$\rho a^2(1 - \Omega^{-1}) = \frac{3k}{8\pi G}. \tag{10}$$

The right hand side of this equation is a constant, the left hand side must therefore also remain constant as the universe evolves. In the time between the infant universe and the present, the term ρa^2 must decrease by many orders of magnitude (remember that $\rho \propto a^{-3}$ and $\rho \propto a^{-4}$ in a matter and radiation dominated universe, respectively). As current measurements suggest $\Omega \approx 1.0$ the above equation would indicate careful tuning of the density parameter. This is the **flatness problem**.

Let's write out the derivation of Equation 10. Starting from Equation 1, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \implies \frac{3\dot{a}^2}{8\pi G} = a^2\rho - \frac{3k}{8\pi G}$$
$$\implies a^2\rho = \frac{3}{8\pi G}(\dot{a}^2 + k) = \frac{3H^2}{8\pi G}a^2 + \frac{3k}{8\pi G}$$
$$\implies a^2(\rho - \rho_{\text{crit}}) = \frac{3k}{8\pi G}$$
$$\implies \rho a^2(1 - \Omega^{-1}) = \frac{3k}{8\pi G}$$

where we have used

$$\rho_{\text{c}} \equiv \frac{3H^2}{8\pi G}. \tag{11}$$

4 Cosmic Inflation

Cosmic inflation was invoked to resolve a number of cosmology peculiarities.[1] The basic idea is that the infant universe undergoes a period of accelerated expansion which is characterized by $\ddot{a} > 0$. This quick expansion of space magnifies microscopic volumes and washes out spatial curvature. As part of this process, regions that were once in causal contact are expanded away from each other at superluminal speeds so that they now appear to be causally disconnected.

At some point, however, this dramatic expansion has to slow down to the more moderate rate that allowed for structure formation. A generic model of inflation is generally followed by a so-called reheating event, during which different particle

[1] I highly recommend Alan Guth's historical accounting of his own work as described in the popular science book, the Inflationary Universe [1].

species are created in thermal equilibrium.

4.1 Horizons and Inflation

By looking at Equation 2, we would be forgiven for assuming that the expansion of the universe is always slowing down,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p).$$

The right-hand side of the above equation is a negative number as long as p and ρ are zero or positive. Accelerated expansion requires the right-hand side to be positive, which in turn means that $\rho + 3p < 0$. Let's poke at this a bit further.

We start by rewriting Equation 7.

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_{a_i}^{a_f} \frac{da}{a\dot{a}} = \int_{a_i}^{a_f} (aH)^{-1} d\ln a. \tag{12}$$

This demonstrates a link between the comoving Hubble radius, $(aH)^{-1}$, and the causal relations in our universe. Remembering that radiation and matter follow equation of states $p = w\rho$ with $w = 1/3$ and $w = 0$, respectively, we can show that

$$(aH)^{-1} = \kappa a^{\frac{1}{2}(1+3w)}, \tag{13}$$

where κ is some overall normalization factor. This result suggests that the comoving Hubble radius is expanding for "normal" forms of energy.

In order to solve the Horizon problem, we want to find some physical mechanism that reverses this process at earlier times. This would mean that all of the observable universe could have been in causal contact at some earlier time. Allowing

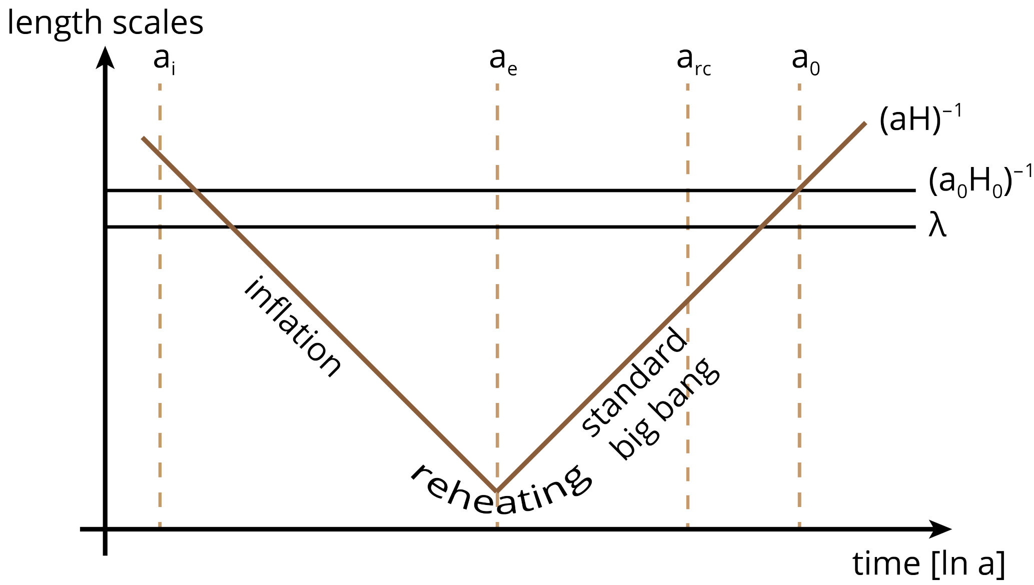

Figure 3: Evolution of length scales as a function of time.

fluctuations to equilibrate before they were separated by normal expansion of the universe.

Finally, we note that

$$\frac{d(aH)^{-1}}{dt} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0 \Rightarrow \ddot{a} > 0. \tag{14}$$

In other words, a shrinking Hubble radius suggests an accelerated expansion of the universe. This is the condition for cosmic inflation.

5 The Inflaton

Scalar fields are widely used in physics. A prominent example is the hypothesized Higgs field whose existence was arguably confirmed with the discovery of the Higgs boson in 2012 [2, 3]. We assume a scalar field that is invariant under translation,

$$\phi(t, \mathbf{x}) = \phi(t). \tag{15}$$

This field can possess both kinetic and potential energy and the pressure and energy density associated with a scalar field are denoted as (see discussion in Section 10.3 in B&G)[2]

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{16}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{17}$$

In the context of cosmology, the field ϕ is normally referred to as the inflaton. The concept of a scalar field is made more clear in an introductory quantum field theory course.

5.1 Inflation Dynamics

Slow-roll inflationary models make simplifying assumptions about the dynamics of a single scalar field. In turn, these allow one to succinctly capture the requirements for inflation and parametrize the corresponding predictions for observables. The following discussion relies heavily on the wonderfully written textbook by Lyth and Liddle [4]. First we use the continuity equation for a perfect fluid:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{18}$$

from which the second Friedmann equation is derived. This allows us to define the equations of motion for the inflaton field:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right], \tag{19}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -V'(\phi). \tag{20}$$

[2] Daniel Baumann's lecture notes on cosmology also discuss the derivation of the scalar field dynamics, see http://www.damtp.cam.ac.uk/user/db275/Cosmology.pdf.

Where the second equation was derived from simply plugging Equations 16 and 17 into Equation 18. Notice the similarity between Equation 20 and that of an equation describing a driven harmonic oscillator. The potential acts as a driving force while the expansion of the universe adds friction.

At this point, its useful to take yet another look at the time derivative of the Hubble radius.

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\epsilon), \quad \text{where} \quad \epsilon \equiv -\frac{\dot{H}}{H^2}.$$

A shrinking Hubble radius therefore means that

$$\epsilon = -\frac{\dot{H}}{H^2} < 1. \tag{21}$$

The parameter ϵ is known as one of the slow-roll parameters for reasons that will become more clear.

We can show that Equations 16 and 17 together with the Friedmann Equations lead to

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{m_{\mathrm{P}}^2 H^2}, \tag{22}$$

where have have defined $m_{\mathrm{P}}^2 \equiv (8\pi G)^{-1}$. Inflation ($\epsilon < 1$) therefore only occurs when the kinetic energy term of the scalar field, $\frac{1}{2}\dot{\phi}^2$, is small relative to its total energy density, $\rho_\phi = 3m_{\mathrm{P}}^2 H^2$.

If the conditions for inflation are supposed to last for a significant amount of time, the acceleration of the scalar field, $\ddot{\phi}$, has to be small relative to $\dot{\phi}$; otherwise the kinetic term of the inflaton will start to catch up with its total energy. In other words, $\ddot{\phi} < H\dot{\phi}$. Defining

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{23}$$

we see that the conditions for sustained inflation are required if $\epsilon, \delta \ll 1$. It is common to define yet another parameter which measures the time development of ϵ according to

$$\zeta = \frac{\dot{\epsilon}}{H\epsilon}. \tag{24}$$

In the limit where $|\zeta| < 1$, the change in ϵ per Hubble time is small. One can show that three parameters that we have introduced are related according to

$$\zeta = 2(\epsilon - \delta). \tag{25}$$

We can now simplify Equations 19 and 20 under the slow-roll assumption:

$$H^2 \simeq \frac{1}{3m_{\mathrm{P}}^2} V(\phi), \tag{26}$$

$$3H\dot{\phi} \simeq -V'(\phi). \tag{27}$$

With all of the above in place, we can show that (this is a useful exercise)

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{m_{\mathrm{P}}^2 H^2} \simeq \frac{1}{2}\left(\frac{V'}{V}\right)^2 m_{\mathrm{P}}^2. \tag{28}$$

and

$$\eta \equiv \epsilon + \delta = -\frac{\dot{H}}{H^2} - \frac{\ddot{\phi}}{H\dot{\phi}} \approx m_{\mathrm{P}}^2 \frac{|V''|}{V}. \tag{29}$$

These two parameters are generally referred to as the slow-roll parameters. Given a hypothetical potential for an inflaton field, we can calculate these slow-roll parameters and estimate if and for how long the requirements for slow-roll inflation are maintained. A typical potential that is often discussed follows

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{30}$$

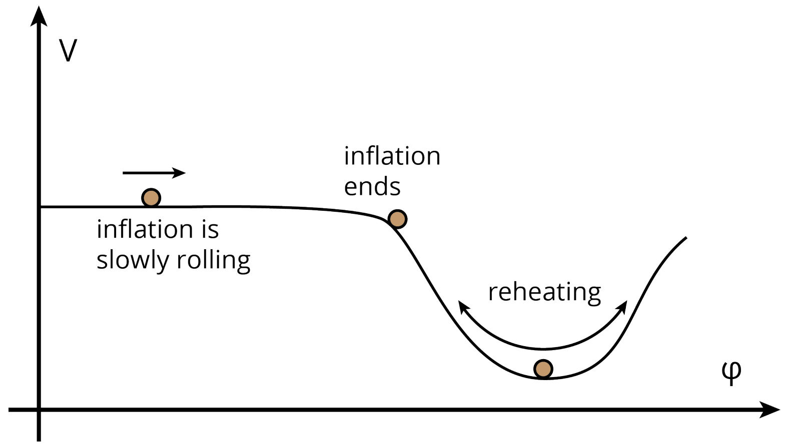

Figure 4: Rough illustration of slow-roll inflation. The inflaton field rolls down its potential to a true ground state.

5.2 Conclusions

We have shown that accelerated expansion requires that the potential energy of the scalar field dominates the kinetic energy for a prolonged time period. Such conditions can arise if the potential is very flat; a scalar field would then roll slowly down the potential (see Figure 4).

In some sense, inflation represents a dynamical cosmological constant. During inflation, regions that were disconnected become causally connected while Ω tends to unity and exotic particles, such as magnetic monopoles, are redshifted to very low densities. Inflation suppresses some of the complications associated with the hot big bang model. It must conclude, however, with the conversion of the inflaton energy density into normal matter during a period called reheating.

The slow-roll parameters, ϵ and δ, encapsulate the simplest set of models that produce smoothing, flattening, and monopole dilution, the basic requirements of the inflationary paradigm.

6 Relation to Cosmological Observables

CMB observables can be conveniently linked to the slow-roll parameters (see Equations 28 and 29). The RMS amplitude of scalar fluctuations has already been well constrained through measurements of the CMB; the temperature anisotropies are incredibly uniform, with only 100 part in million fluctuations. Based on this, we write $\mathcal{P}_{\rm S}(k_0) \sim 100 \times 10^{-6}$, where $\mathcal{P}_{\rm S}$ is the scalar perturbation power spectrum and k_0 corresponds to the scale of the measurement. Having established this measurement, the energy scale of inflation within the slow-roll scenario can be expressed as

$$V^{1/4} = \epsilon^{1/4}\, 9 \times 10^{16} \mathrm{GeV}. \tag{31}$$

Similarly, the spectral index, $n_{\rm S} - 1 = d \ln \mathcal{P}_{\rm S} / d \ln k$, describing the scale dependence of the curvature perturbations follows

$$n_{\rm S} = 2\eta - 6\epsilon. \tag{32}$$

A measurement of primordial B-modes constrains the amplitude of tensor perturbations, $\mathcal{P}_{\rm T}$, and therefore the so-called tensor-to-scalar ratio, r, which in turn is related to one of the slow-roll parameters:

$$r \equiv \frac{\mathcal{P}_{\rm T}(k_0)}{\mathcal{P}_{\rm S}(k_0)} = 16\epsilon. \tag{33}$$

Finally, the spectral tilt of the tensor power spectrum can be succinctly written as $n_T = -2\epsilon$. Under the assumption that slow-roll inflation accurately captures the dynamics of the early universe, a measurement of the tensor-to-scalar ratio constrains the energy scale of inflation.

A great number of inflationary models have been proposed. Although many of these models fall under the umbrella of single field slow-roll inflation, there seem to be no limits to the complexity that inflationary models can assume. It should also be pointed out that the inflationary paradigm is by no means the only viable physical model that describes the early universe [5].

Additional Comments and Further Reading

The idea behind cosmic inflation was conceived by Guth while studying the magnetic monopole problem, yet another complication involving conditions of the early universe.

Some particle physics theories predict a number of "relics" from the very hot radiation era, at $T_{\mathrm{GUT}} \sim 10^{15}\,\mathrm{GeV}$ magnetic monopoles should have been produced. Also string theory predicts new fields at these energies. These very massive particles should dominate the energy density of the universe, yet they have not been observed. How did they all vanish? This is the monopole problem.

Guth found that a scenario involving rapid expansion of the universe, sourced by a scalar field, could reduce the density of magnetic monopoles to present-day densities; magnetic monopoles have so far eluded detection. Soon thereafter Guth realized that inflation would also resolve the horizon and flatness problems [1, 6]. The theory of inflation quickly gained momentum, with solutions to outstanding problems proposed by Albrecht, Linde, Steinhardt, and others [7–9].

A plethora of textbooks and review articles describe the concepts briefly reviewed in this text. Some of the more common textbooks include [4, 10]. Review articles that describe cosmology theory in broad strokes include [11–13], while texts like [14] delve into the gory details of cosmic perturbation theory. There also exist review articles highlighting some of the differences between the various experiments involved in observational cosmology [15].

References

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