Lecture 13

FK5024: Particle and Nuclear Physics, Astrophysics and Cosmology

PART III: Astrophysics and Cosmology

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Lecture 12 introduced some fundamental concepts in cosmology. In this lecture we will expand a bit on some of those concepts. In particular, we will focus on Hubble expansion and role of the Friedmann in describing the time-evolution of the Universe.

To be supplemented by Liddle: 3.4-3.5, 4 (skip 4.4-4.5), 5.3-5.5

1 Hubble expansion

The Universe is expanding. Galaxies tend to move away from us. Remembering the notation from Lecture 12, we write:

$$r = a\chi, \tag{1}$$

where r is the physical distance, a is the scale factor, and χ is the comoving distance. We can then write:

$$v(t) = \frac{dr(t)}{dt} = \frac{d}{dt}\left[a(t)\chi\right] = \dot{a}\chi = \frac{\dot{a}}{a}r. \tag{2}$$

In this expression $H(t) = \dot{a}/a$ is the Hubble parameter. Evaluated at the current epoch, we write $t = t_0 \Rightarrow H(t_0) \equiv H_0 = \dot{a}(t_0)/a(t_0)$. The Hubble constant H_0 is the Hubble parameter, $H(t)$, evaluated at the current epoch $t = t_0$.

Observations suggest that

$$H_0 \approx (70 \pm 3)\,\mathrm{km/s/Mpc}. \tag{3}$$

Other conventions include writing $H_0 \equiv h \cdot 100\,\mathrm{km/s/Mpc}$, with $h = 0.70 \pm 0.03$ (see formula sheet).

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The low-velocity limit of the relativistic Doppler equation gives

$$\frac{\lambda_0}{\lambda_e} = 1 + \frac{v}{c} = \frac{a(t_0)}{a(t_e)} = 1 + z \tag{4}$$

where λ_e and λ_0 are the wavelength of the radiation when it was emitted and when it was observed, respectively. For objects that are relatively close to us, the scale factor at the time of emission will be similar to $a(t_0)$. In that case, peculiar motion of the emitting object relative to us becomes important. For very large cosmological distances, $z > 1$, the approximations needed for the above expression start to break down.

At this point, it might be helpful to review distance scales

- Distance to sun: 4.84×10^{-12} Mpc
- Distance to Alpha Centauri: 1.33×10^{-6} Mpc
- Distance to center of Milky Way: 0.008 Mpc
- Distance to Andromeda (M31): 0.778 Mpc
- Distance to the Virgo galaxy cluster: 17 Mpc
- Distance to the surface of last scattering (CMB): $13,700$ Mpc

Note that the Virgo Cluster is a cluster of about 1000-2000 galaxies. The cluster appears to be at the centre of a larger supercluster, of which the Local Group (a group of galaxies that includes the Milky Way) is a member.

2 Friedmann Equations from Newtonian Mechanics

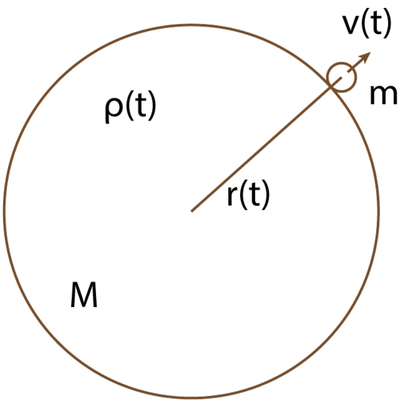

Figure 1: Homogenous sphere.

The Friedmann equations describe the time-evolution of an isotropic and homogenous universe that conforms to Einstein's theory of general relativity.

It turns out that Newtonian physics can be used to derive the Friedmann equations.[2] Assuming a homogenous sphere of density $\rho(t)$ and radius $r(t)$, how will the velocity of a test mass particle with mass m, evolve with time (see Figure 1)?

We can first note that the mass within a radius r is

$$M(t) = \frac{4\pi}{3} r^3(t)\rho(t). \tag{5}$$

[2]This section is based on discussion in Liddle 3.1.

The potential energy of a test particle moving in this matter distribution is

$$V(T) = -\frac{GM(t)m}{r(t)} + \text{const} = -\frac{4\pi G}{3} r^2(t)\rho(t)m. \tag{6}$$

The kinetic energy of our test particle is

$$T(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}m\dot{r}^2. \tag{7}$$

Combining the above with $r(t) = a(t)\chi$ and $\dot{r} = \dot{a}\chi$ we find:

$$U = T + V = \frac{1}{2}m\dot{a}^2\chi^2 - \frac{4\pi G}{3}a^2\chi^2\rho(t)m, \tag{8}$$

$$= m(a\chi)^2 \left[\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{4\pi G}{3}\rho\right]. \tag{9}$$

Rearranging the above we get:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{m(a\chi)^2},$$

$$= \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a^2}. \tag{10}$$

We define $k \equiv -2U/(mc^2\chi^2)$, which is a constant because energy is conserved and the comoving distance is fixed. This is the so-called 1st Friedmann equation.

Note that k is constant with time. The Universe has a unique value for k which is unchanged with time. Also note that a positive k implies a negative U. This implies that the expansion of the Universe will sometime come to a halt and reverse itself. Conversely, a negative value for k implies a positive value for U. In this case the Universe will expand forever.

> **Note.** The above derivation relies on the Shell theorem (which is related to Birkhoff's theorem) from classical mechanics, which states that a spherically symmetric body affects external objects gravitationally as if all of the mass were concentrated at a point at its centre. This is also strongly related to Gauss's law for gravity which has a direct analogy to Gauss's law in electromagnetism.

3 Fluid equation

The 1st Friedmann equation in isolation lacks a bit of punch. We want to better understand how the properties of different energy components (and their density,

$\rho(t)$) influence their time evolution. This behaviour is encapsulated in the so-called fluid equation. We can derive this equation from the 1st law of thermodynamics

$$dE + pdV = TdS, \tag{11}$$

where V now corresponds to an expanding volume of comoving radius. The above equation is simply a statement of local energy conservation. Imagine that we are observing gas confined to a sphere of unit comoving radius. The physical radius at any time can be found from a and we can use $E = mc^2$ to write

$$E = \frac{4\pi}{3} a^3 \rho c^2. \tag{12}$$

Taking the total time derivative, we find that

$$\frac{dE}{dt} = 4\pi a^2 \rho c^2 \frac{da}{dt} + \frac{4\pi}{3} a^3 \frac{d\rho}{dt} c^2, \tag{13}$$

and the volume rate of change is

$$\frac{dV}{dt} = 4\pi a^2 \frac{da}{dt}. \tag{14}$$

Assuming an adiabatic expansion (isolated and reversible process) $dS = 0$, we are left with $dE = -pdV$ and

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{15}$$

This is the fluid equation that we will use to describe the time evolution of different energy components in the Universe.

4 Acceleration equation

The 2nd Friedmann equation, also known as the acceleration equation, can be derived from the 1st Friedmann equation and the fluid equation. We start by differentiating the 1st Friedmann equation (Equation 10) w.r.t. to time.

$$2\frac{\dot{a}}{a}\frac{a\ddot{a} - \dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{kc^2\dot{a}}{a^3}. \tag{16}$$

Inserting Equation 15 into the above we find

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2}. \tag{17}$$

Then, plugging in Equation 10 a second time, we find:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right). \tag{18}$$

This equation shows that any energy component with a pressure term tends to slow down the expansion; the second time derivative of the scale factor will be negative.

5 Units

Up to this point we have been careful about including the speed of light, c, explicitly in our derivations. However, it is common to set $c = 1$. The physics are not changed by this fact and the equations are simplified. However, this can also lead to issues when trying to quote physically meaningful quantities.

Using units where we set $c = 1$, the 1st and 2nd Friedmann equations become:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}} \tag{19}$$

and

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right).} \tag{20}$$

6 The equation of state

A famous equation of state is the gas equation

$$pV = nRT. \tag{21}$$

The gas equation allows us to relate gas pressure to other state variables such as number density, n, and temperature, T. In cosmology, it is common to express matter/energy equations of state as simply

$$p = w\rho, \tag{22}$$

where p is pressure, ρ is energy density, and w is some scalar that relates the two.

For ideal gas the gas equation tells us that

$$pV = \frac{1}{3}Nmv_{\rm rms}^2. \tag{23}$$

Observations show that the average peculiar velocity of galaxies are well within relativistic limits ($v/c << 1$) and we do in fact assume that all matter is non-relativistic.[3] This implies that the pressure term in the acceleration equation can be ignored, that is $p/c^2 \simeq 0$. In this case, the fluid equation says that

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad (p = 0),$$
$$\Rightarrow \quad \frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a},$$
$$\Rightarrow \quad \ln(\rho) = -3\ln(a) + \text{ const} \tag{24}$$

From this we can write

$$\rho = \rho_0 a^{-3}. \tag{25}$$

In other words

$$\boxed{\rho_\text{m} \propto \frac{1}{a^3}} \tag{26}$$

What happens if $p \neq 0$? If we assume that one can write $p = w\rho$, we will get

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\left(\frac{\dot{a}}{a}\right)$$
$$\Rightarrow \rho \propto \frac{1}{a^{3(1+w)}} \tag{27}$$

Photons have a pressure term that is equal to 1/3 the energy density (see statistical mechanics course). In that case, we arrive at

$$\boxed{\rho_\text{r} \propto \frac{1}{a^{3(1+\frac{1}{3})}} = \frac{1}{a^4}} \tag{28}$$

This suggests that the energy density of radiation and matter evolve differently with change in the scale factor. In the case of photons, a more rapid decline in the energy density with an expanding universe can be understood in terms of a wavelength increase.

> **Note.** Pressure is typically quoted in units of N/m^2 (Newton per meter squared). However, multiply both sides of the division symbol and you will find that pressure can be expressed as Nm/m^3 which has the same units as energy density since J = Nm.

Remember that $E = hf = hc/\lambda$, a change in the wavelength because of the expansion of space therefore reduces the energy of photons; the photons are redshifted.

[3] For example, the velocity of our galaxy relative to the fixed reference frame defined by the CMB photons (which we will cover in a future lecture) is about $\mathcal{O}(100)$ km/s.

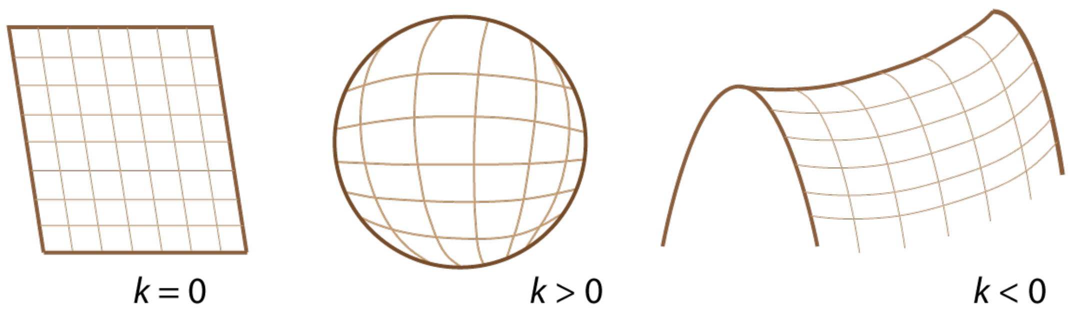

Figure 2: Two-dimensional analog of universe curvature.

7 Curvature

Up to this point, we have treated the parameter k as a simple constant. It turns out, however, that k has physical relevance. In short, general relativity permits three types of solutions to the equations describing the time evolution of our Universe when assuming homogenity and isotropy. These are attributed to three types of values of the constant k, $k < 0$, $k = 0$, and $k > 0$.

$\boldsymbol{k = 0}$ This is the most intuitive case. A value of $k = 0$ corresponds to a flat universe that obeys Euclidian geometry. Two parallel lines will remain parallel indefinitely. This universe is infinite in extent, since some type of an edge would break our requirement of homogeneity and isotropy. Figure 2 gives a visual representation of a flat universe in two dimensions. The sum of angles in a triangle is 180 degrees.

$\boldsymbol{k > 0}$ This is a closed universe with parallel lines eventually crossing each other (positive curvature); this universe is finite in extent. The two-dimensional analogy is simply the surface of a sphere. Note that the 2-dimensional analogy is still perfectly consistent with the concept of isotropy and homogeneity. Triangles drawn in this geometry will have sum of angles that are larger than 180 degrees.

$\boldsymbol{k < 0}$ This is an open universe (negative curvature). Lines that are parallel at some point will diverge and never cross. This universe is infinite in extent. Triangles drawn in this geometry will have sum of angles that are less than 180 degrees.

Why do we care about curvature? Although it is conceptually easy to think of a flat universe, the theory of general relativity allows for three types of solutions and we have no reason to assume one over the other.

8 The case of a flat universe

In a flat ($k = 0$) and matter dominated universe we can write

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t). \tag{29}$$

From equation 25 we saw that one can write

$$\rho_{\rm m}(t) = \frac{\rho_{\rm m}(t_0)}{a^3(t)} \tag{30}$$

where we have chosen $a(t_0) = 1$. Multiplying through Equation 29 with $a(t)^3$ we get

$$\dot{a}^2 a = \frac{8\pi G}{3}\rho_0 = \text{constant} \tag{31}$$

Assuming one can write $a(t) = t^q$ we have

$$\dot{a} = \frac{1}{q}t^{q-1}, \tag{32}$$

and therefore

$$\begin{aligned} \dot{a}^2 a &= t^{2(q-1)}t^q = t_0 = \text{constant} \\ &\Rightarrow 2(q-1) + q = 0 \\ &\Rightarrow q = \frac{2}{3}. \end{aligned} \tag{33}$$

This tells us that in a flat and matter dominated universe the scale factor, $a(t)$, evolves like

$$a(t) \propto t^{2/3} \quad \text{(matter dominated)}. \tag{34}$$

A similar exercise for a flat, but radiation dominated universe gives

$$a(t) \propto t^{1/2} \quad \text{(radiation dominated)}. \tag{35}$$

9 Age of the universe

What is the age of a matter dominated universe? Assuming a matter dominated universe at all times ($\rho_m \gg \rho_r$), we have

$$a \propto t^{2/3}. \tag{36}$$

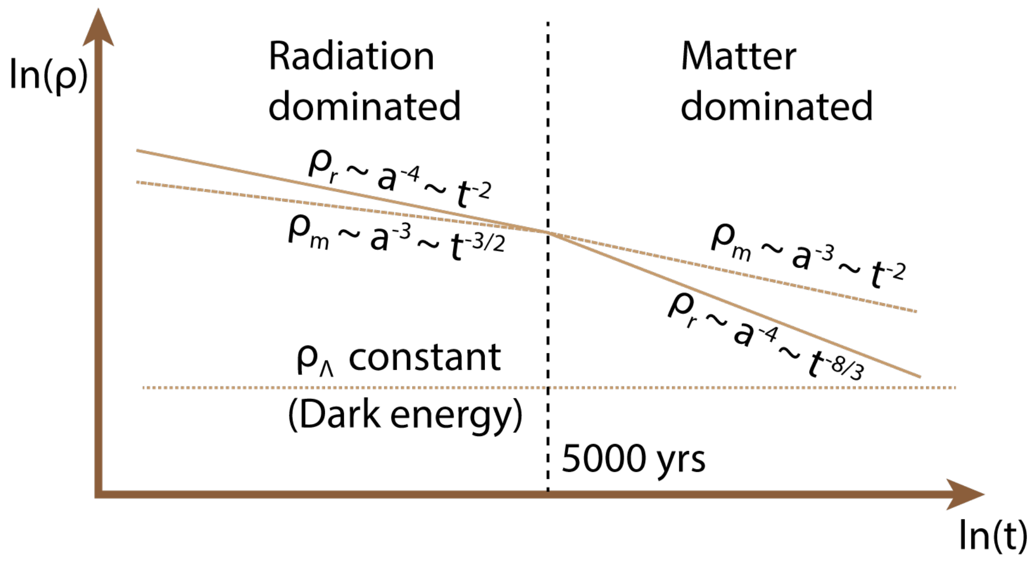

Figure 3: Time evolution of different energy components.

From this, we find that

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3}\frac{t^{-1/3}}{t^{2/3}} = \frac{2}{3t}. \tag{37}$$

Which allows us to write

$$t_0 = \frac{2}{3H_0} = \frac{2}{3 \cdot 2.3 \times 10^{-18}\,\mathrm{s}} = 3 \cdot 10^{17}\,\mathrm{s} \approx 10^{10}\,\mathrm{years}, \tag{38}$$

noting that $H_0 = 70\,\mathrm{km/s/Mpc} \approx 2.3 \times 10^{-18}\,\mathrm{s}^{-1}$.

Our current best estimates of the age of the universe (see future lecture) puts it at

$$T_{\mathrm{age}} = 13.7 \times 10^9\,\mathrm{years}. \tag{39}$$

10 Time evolution of the universe

At the present epoch, radiation appears to provide a relatively small fraction of the total energy density of the universe. In fact, $\rho_{\mathrm{rad}}/\rho_{\mathrm{matter}} \sim 10^{-5}$. However, the Friedmann equations (and Equations 26 and 28) show the relative energy contribution of radiation and matter is not fixed with time. At earlier times

$$\frac{\rho_r}{\rho_m} \sim \frac{1/a^4}{1/a^3} \sim \frac{1}{a}. \tag{40}$$

At earlier times, $a \to 0$, the universe was radiation-dominated. Figure 3 shows the time-development of energy density from matter and radiation. At some point in the history of the universe, there was a transition where matter energy density overtook the photon energy density in driving the expansion of space. It turns out that this happened approximately 5000 years after the Big Bang.