Lecture 14

FK5024: Particle and Nuclear Physics, Astrophysics and Cosmology

PART III: Astrophysics and Cosmology

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This lecture will focus on some features of a universe that is described by the Friedmann equations. We will introduce the concept of dark energy (cosmological constant) as well as the so-called critical density. We will also look at a model that combines energy contributions from multiple sources. We will discuss current best estimates for the total energy budget of the universe.

This lecture should be supplemented by Liddle: 6-9

1 The cosmological constant

In the early 20th century—even in the presence of Hubble's observations—Einstein and the majority of astronomy and physics communities were against the concept of an expanding universe.[2] To combat this, Einstein famously implemented a cosmological constant. With this additional term, the 1st Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{1}$$

In a universe that accommodates matter, radiation, and a cosmological constant, the acceleration equation becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_{i\in[\mathrm{m,r,\Lambda}]} (\rho_i + 3p_i) = -\frac{4\pi G}{3} \sum_{i\in[\mathrm{m,r,\Lambda}]} \rho_i(1 + 3w_i), \tag{2}$$

where we assume that there's an equation of state that allows one to write

$$p_i = w_i \rho_i. \tag{3}$$

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[2] There must be dozens of books discussing this period in the development of cosmology. One book that I particularly liked describes the role of Geogre Lemaitre in this story [1].

Note that m, r, and Λ, correspond to matter, radiation, and a cosmological constant, respectively.

> **Note.** For most purposes and in most scientific discussion, dark matter and ordinary baryonic matter are treated separately. However, when it comes to describing the equation of state of these two constituents, they are treated equally. Both dark matter and ordinary matter are treated as a pressureless component ($w = 0$). This is consistent with a picture where dark matter particles correspond to unidentified non-relativistic particles (cold dark matter).

Einstein invoked the cosmological constant to balance the contributions of energy density, ρ, curvature, k, and the cosmological constant, Λ, so that we would get

$$H(t) = 0. \tag{4}$$

This turns out to be quite difficult in the presence of the two Friedmann equations, since any solution that gives $H(t) = 0$ is unstable to perturbations.

It is instructive to look at the acceleration equation in the presence of a cosmological constant. It becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{5}$$

If a positive cosmological constant term dominates the first term, it will induce a positive value for $\ddot{a}$. In other words, a relatively large and positive cosmological constant will lead to accelerated expansion.

The modern interpretation is that Λ represents vacuum energy; the idea that empty space has an intrinsic energy density. Figure 1 shows the qualitative difference between a classical vacuum and one that is full of pseudo-particles that are popping in and out of existence.

When attempting to calculate the expected value for the energy density of vacuum, high-energy physicists (or particle physicists) find that the expected value is in the range of 60-120 orders of magnitude greater than what is suggested by current cosmological limits on magnitude of the cosmological constant. This is known as the cosmological constant problem or the vacuum catastrophe.

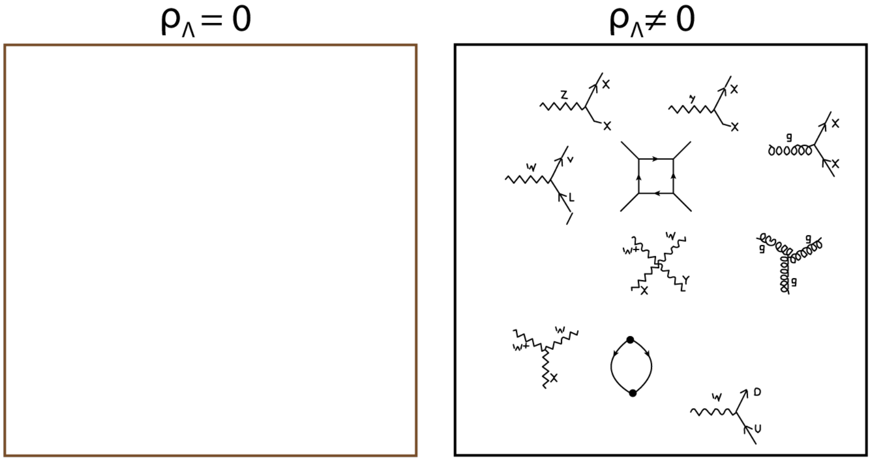

Figure 1: Vacuum fluctuations in "empty space" compared to classical empty space. Particles are popping in and out of existence leading to non-zero average energy of empty space.

1.1 Equation of state for cosmological constant

Let's define

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \quad \text{(constant)}. \tag{6}$$

This gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_\mathrm{m} + \rho_\mathrm{r} + \rho_\Lambda) - \frac{k}{a^2}. \tag{7}$$

The requirement that ρ_Λ is constant implies that $\dot{\rho}_\Lambda = 0$. We can now use the fluid equation to derive the equation of state for dark energy.

$$\dot{\rho}_\Lambda + 3\left(\frac{\dot{a}}{a}\right)(\rho_\Lambda + p_\Lambda) = 0$$
$$\Rightarrow \rho_\Lambda + p_\Lambda = 0. \tag{8}$$

In other words $\rho_\Lambda = -p_\Lambda$. Using our standard notation, $p = w\rho$, we get $w = -1$. If the energy density is positive we therefore must have negative pressure!

2 Critical density

Examining the 1st Friedmann equation, we see that there is a value for the total energy density, ρ that forces $k = 0$. This density is

$$\rho_c \equiv \frac{3H^2(t)}{8\pi G}. \tag{9}$$

Evaluated at the current epoch $t = t_0$, this number is

$$\rho_c(t_0) = \frac{3H_0^2}{8\pi G} = 1.88 \times 10^{-26}\, h^2 \frac{\mathrm{kg}}{\mathrm{m}^3}. \tag{10}$$

Note that mass of the proton is $1.67 \times 10^{-27}\,\mathrm{kg}$. Assuming $h = 0.7$, the critical density then corresponds to approximately 5-6 protons per cubic meter! This sounds like a really small number. However, when we look at cosmological distances this does not sound as strange.

Remembering that $G = 6.67 \times 10{-}11\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$, $M_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$, and $1\,\mathrm{Mpc} = 3.09 \times 10^{22}\,\mathrm{m}^3$ we get

$$\rho_\mathrm{c} = 2.8h^{-1} \times 10^{11} m_\odot/(h^{-1}\,\mathrm{Mpc})^3. \tag{11}$$

Note that mass of the Milky Way is estimated at approximately $1 \times 10^{12}\, m_\odot$ and the typical distance between galaxies is of order $1\,\mathrm{Mpc}$.

It is standard to define the density parameter relative to the critical density

$$\Omega(t) = \frac{\rho(t)}{\rho_\mathrm{c}(t)} = \frac{\rho_\mathrm{m}(t) + \rho_\mathrm{r}(t)}{\rho_\mathrm{c}(t)} = \Omega_\mathrm{m}(t) + \Omega_\mathrm{r}(t). \tag{12}$$

We sometimes drop the time-dependence (leave it implicit). In this case, the 1st Friedmann equation for a universe that includes both matter and radiation becomes

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho_\mathrm{c}\Omega - \frac{k}{a^2}, \\ &= H^2\Omega - \frac{k}{a^2} \end{aligned} \tag{13}$$

where $\Omega = \Omega_\mathrm{r} + \Omega_\mathrm{m}$ and we can rearrange to find

$$\Omega - 1 = \frac{k}{a^2H^2}. \tag{14}$$

Note that $\Omega = 1$ is a special case since k is a constant. It implies that $\Omega = 1$ at all times. After defining

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2(t)}, \tag{15}$$

and

$$\Omega_k \equiv -\frac{k}{H^2(t)a^2(t)}, \tag{16}$$

we can also rewrite the above equation to get

$$\Omega + \Omega_k = 1. \tag{17}$$

If we're including a cosmological constant, we would get

$$\Omega_{\rm r} + \Omega_{\rm m} + \Omega_\Lambda + \Omega_k = 1. \tag{18}$$

In a flat universe ($k = 0$),

$$\Omega_{\rm r} + \Omega_{\rm m} + \Omega_\Lambda = 1. \tag{19}$$

The time-evolution of different energy components is coupled.

3 Total energy budget

We can put all of this together to find that

$$\frac{k}{a^2H^2} = \Omega_{\rm r} + \Omega_{\rm m} + \Omega_\Lambda - 1. \tag{20}$$

The curvature of the universe depends on the fractional contribution of different energy components. We find that:

- Open universe: $k < 0$ and $\Omega_{\rm r} + \Omega_{\rm m} + \Omega_\Lambda < 1$
- Flat universe: $k = 0$ and $\Omega_{\rm r} + \Omega_{\rm m} + \Omega_\Lambda = 1$
- Closed universe: $k > 0$ and $\Omega_{\rm r} + \Omega_{\rm m} + \Omega_\Lambda > 1$

We can define

$$\Omega_{\rm m,0} \equiv \Omega_{\rm m}(t = t_0), \quad \Omega_{\rm r,0} \equiv \Omega_{\rm r}(t = t_0), \quad \Omega_{\Lambda,0} \equiv \Omega_\Lambda(t = t_0), \tag{21}$$

which allows us to write

$$H^2(t) = H_0^2 \left[\Omega_{\rm m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{\rm r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_k\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0}\right]. \tag{22}$$

This is sometimes written in terms of redshift

$$H^2(z) = H_0^2 \left[\Omega_{\rm m,0}(1+z)^3 + \Omega_{\rm r,0}(1+z)^4 + \Omega_k(1+z)^2 + \Omega_{\Lambda,0}\right]. \tag{23}$$

Current best estimates suggest that $\Omega_{\rm m,0} \sim 0.3$, $\Omega_{\rm r,0} \sim 1 \times 10^{-5}$, $\Omega_{\rm k} \sim 1 \times 10^{-3}$, $\Omega_{\Lambda,0} \sim 0.7$. In other words, at the present epoch it would appear that matter and dark energy dominate. Furthermore, recent observations from the *Planck* satellite limit the curvature of the Universe to

$$\Omega_{\rm k} = 0.001 \pm 0.002. \tag{24}$$

In other words, current best estimates suggest that the universe if flat.

Figure 2 shows a selection of time evolution histories of the scale factor for different values of the total universe energy density.

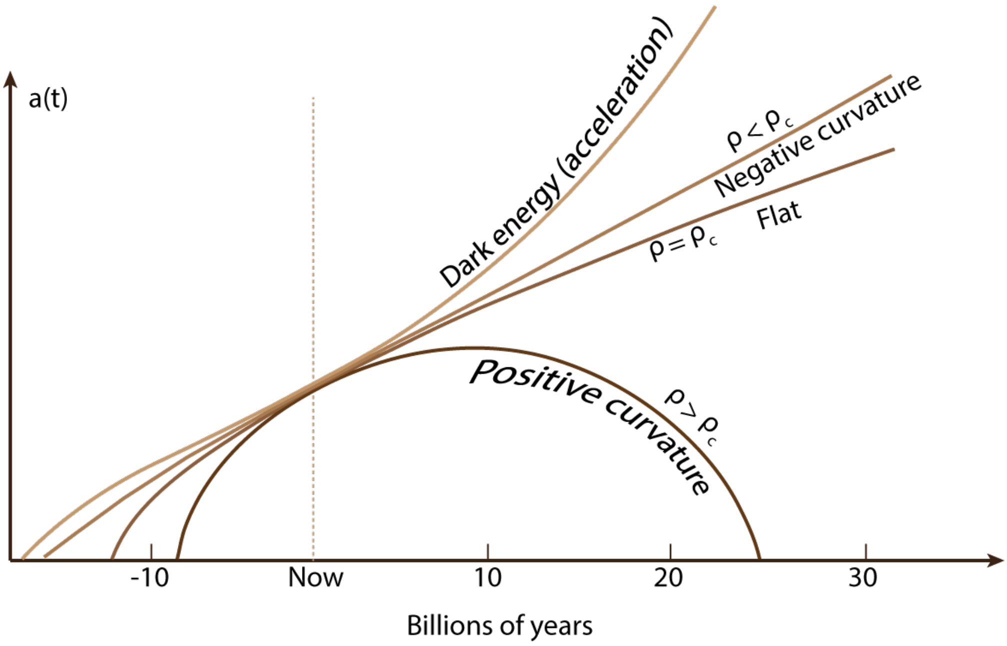

Figure 2: Scale factor as a function of time for different total energy densities. Figure adapted from a similar figure found in Dr. Neil Trentham's lecture notes.

4 Deceleration/acceleration parameter

We can Taylor expand the scale factor around $t = t_0$.

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a}(t_0)(t - t_0)^2 + \dots \tag{25}$$

Dividing through:

$$\begin{aligned} a(t) &= a(t_0)\left[1 + \frac{\dot{a}(t_0)}{a(t_0)}(t - t_0) + \frac{1}{2}\frac{\ddot{a}(t_0)}{a(t_0)}(t - t_0)^2 + \dots\right] \\ &\equiv a(t_0)\left[1 + H(t_0)(t - t_0) - \frac{q_0}{2}H_0^2(t - t_0)^2 + \dots\right], \end{aligned} \tag{26}$$

where

$$q_0 \equiv \frac{-\ddot{a}(t_0)}{a(t_0)H_0^2} = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}^2(t_0)} \tag{27}$$

is the (dimensionless) deceleration parameter.

Separate energy components evolve differently with the scale factor and from the acceleration equation we find

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i \rho_i(1+3w_i) \tag{28}$$

$$\Rightarrow q_0 = \frac{-\ddot{a}(t_0)}{H_0^2 a(t_0)} = \frac{4\pi G}{3H_0^2}\sum_i \rho_i(1+3w_i) \tag{29}$$

Noting that $4\pi G/(3H_0^2)$ is a constant we see that "fluids" with a positive value for w will decelerate the rate of expansion. Dark energy on the other hand, tends to accelerate the rate of expansion.

5 Connection to exam

It can be useful to connect the lecture material to types of questions that might possible pop up on an exam. Here's a typical exam question:

Question: We assume that Type Ia variables are standard candles and therefore that their luminosity is fixed, $L = L_{\rm sn}$ (see Figure 3). We observe a supernova in a satellite galaxy of the Milky Way (our own galaxy) and thankfully we also have observations of cepheid variables that help us determine the distance to this supernova, R_0. The flux from this supernova is F_0. We now detect a supernova in what appears to be a distant galaxy. The relative magnitude of this supernova compared to close-by supernova is $\Delta m = +9$. Assuming that the luminosity of these two stellar remnants is the same. Derive an equation that describes the distance to the far-away supernova in terms of R_0 and z for **a)** a universe that is static; **b)** in a universe that is expanding such that $a(t_0)/a(t) = 1 + z$ where $a(t)$ is the scale factor and z is the redshift.

Answer: a) From lecture 11, in a static universe we know that the apparent magnitude can be written as

$$m_0 = -2.5\log_{10}\left(\frac{F_0}{F_{\rm ref}}\right) + {\rm const.} \tag{30}$$

is the magnitude of the nearby supernova. The magnitude of the distant supernova is

$$m_1 = -2.5\log_{10}\left(\frac{F_1}{F_{\rm ref}}\right) + {\rm const.} \tag{31}$$

The relative magnitude of the distant supernova compared to the nearby one is

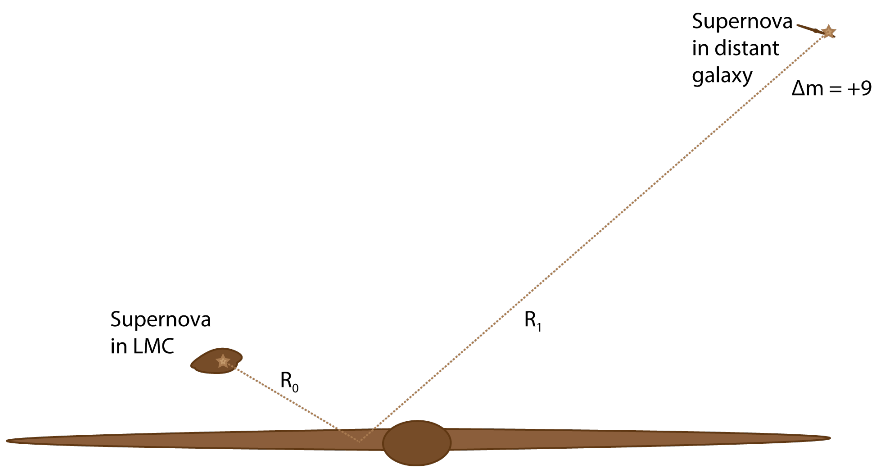

Figure 3: Problem setup.

therefore

$$\Delta m = m_1 - m_0 = -2.5\log_{10}\left(\frac{F_1}{F_0}\right) = 9.0 \tag{32}$$

Using the relation between luminosity and flux, $F \propto R^{-2}$ we see that

$$m_1 = -5\log_{10}\left(\frac{R_{\rm ref}}{R_1}\right) + {\rm const.} \tag{33}$$

which allows us to write

$$\Delta m = \log_{10}\left(\frac{R_0}{R_1}\right) = -9/5 \tag{34}$$

$$\Rightarrow R_1 = R_0/10^{-9/5} \tag{35}$$

$$\Rightarrow R_1 \approx 63.1 R_0 \tag{36}$$

b) In a universe that is expanding we will have to change the luminosity relation. Instead of $F = L/(4\pi R^2)$ we now have

$$F = \frac{L}{4\pi R^2 (1+z)^2}. \tag{37}$$

Plugging into Equation 33, and assuming that the redshift of the reference supernova is $z \approx 0.0$, we get

$$m_1 = -5\log_{10}\left(\frac{R_1(1+z)}{R_{\rm ref}}\right) + {\rm const.} \tag{38}$$

So

$$R_1 = 10^{\Delta m/5} R_0/(1+z). \tag{39}$$

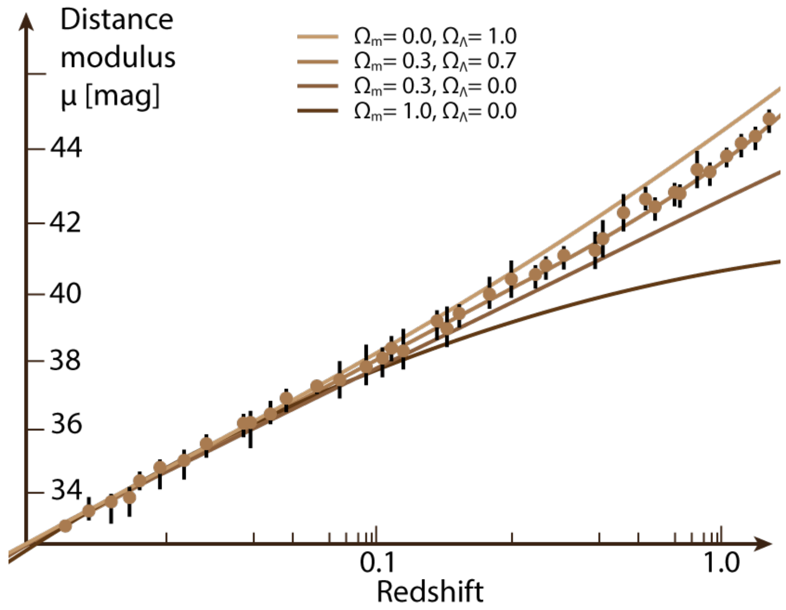

Figure 4: The absolute magnitude (or distance modulus) as a function of redshift for a collection of Type Ia observations. Unlike in the previous figure, we are now looking backwards in time as we move to the right on the graph. Measurements appear consistent with a positive second derivative in the slope which would indicate accelerated expansion.

In an expanding universe, we find that physical distance to the supernova is less than we would assume from a static universe. The photons lose energy because of redshift.

6 Recent observations of accelerated expansion

Let's now connect the above question to today's lecture notes.

Recent observations of Type Ia supernova suggest that the Universe is expanding. This observation led to the awarding of a Nobel prize in 2011. In effect, the supernova measurements show that the deceleration parameter is positive; that the scale factor has a positive second time derivative.

Figure 4 shows a cartoon version of the type of Hubble diagrams that are used to support this claim. Data for hundreds of Type Ia supernova, which we assume are standard candles, have been gathered. These supernovae are found in galaxies covering redshifts up to about $z \approx 1.5$. The distance modulus, or the absolute magnitude, is plotted as a function of redshift. Different cosmological models are shown for comparison. These models are generated using numerical calculations that allow us to evolve the scale factor for different energy density scenarios. It would appear that models with a positive 2nd derivative (acceleration) fit the date points more accurately. In particular, models with $\Omega_\mathrm{m} \approx 0.3$ and $\Omega_\Lambda \approx 0.7$.

References

[1] Farrell, J. **The Day Without Yesterday**. Thunder's Mouth Press, New York, NY, 2005.